An Efficient and Exact Algorithm for Locally h-Clique Densest Subgraph Discovery

Anonymous Author(s)

ABSTRACT

Detecting locally non-overlapping, near-clique densest subgraphs is a crucial problem for community search in social networks. As a vertex may be involved in multiple overlapped local cliques, such as family, office, and laboratory, detecting locally densest sub-structures considering h-clique density, i.e., *locally h-clique densest subgraph (LhCDS)* attracts great interests. This paper investigates the LhCDS detection problem and proposes an efficient and exact algorithm to list the top-k non-overlapping, locally h-clique dense, and compact subgraphs. We in particular jointly consider h-clique compact number and LhCDS and design a new "Iterative Propose-Prune-and-Verify" pipeline (IPPV) for top-k LhCDS detection. (1) In the proposal part, we derive the initial bounds for h-clique compact numbers; prove the validity, and extend a convex programming method to tighten the bounds for proposing LhCDS candidates without missing. (2) Then a tentative graph decomposition method is proposed to solve the challenging case when a clique spans multiple subgraphs in graph decomposition. (3) To deal with the verification difficulty, a basic and a fast verification method are proposed, where the fast method constructs a small-scale flow network to improve efficiency while preserving verification correctness. The verified LhCDSes are output, and the candidates that remained unclear will reenter the IPPV pipeline. (4) We further extend the proposed methods to locally more general pattern densest subgraph detection problems. We prove the exactness and low complexity of the proposed algorithm. Extensive experiments on real datasets show the effectiveness and high efficiency of IPPV.

1 INTRODUCTION

Finding dense subgraphs can uncover highly connected and cohesive structures in graphs, making it an effective tool for understanding complex systems. The discovery of dense subgraphs and communities has numerous applications in diverse fields including social networks [7, 11, 36], web analysis [1, 12], graph databases [16, 39], and biology [21, 30]. In these applications, the identification of near-clique subgraphs holds significant importance, as it relaxes the requirement of complete connectivity within cliques and allows for a certain degree of sparsity or missing connections while still maintaining a high connectivity level.

For the importance of detecting large near-clique subgraphs [35], the h-clique densest subgraph (CDS) problem that finds near-clique graphs formed by overlapped cliques has attracted great research attention [9, 25, 33, 35]. This is due to the fact that a vertex is generally involved in multiple overlapped cliques, such as a person may be involved in cliques as family, office, laboratory, etc. By finding the subgraph with the highest density of h-cliques, CDS uncovers the highly connected component that exhibits strong internal interactions [3, 22, 32]. An example is to discover the most active research group in which the researchers are forming different mutual collaborating groups [26]. Whereas, in the context of the real world, the discovery of a single CDS offers limited insights. Listing the top-k CDSes is desired, but due to the substantial overlap inherent in h-cliques [38], the top-k CDSes may refer to the same dense region, still providing limited structural insights.

Therefore, detecting the top-k non-overlapping, locally maximal, dense, and compact, i.e., *locally h-clique densest subgraphs (LhCDS)* attracts great interest. However, no efficient and exact algorithm is known for detecting LhCDS yet. The closest work to LhCDS discovery is the locally densest subgraph (LDS) discovery [28], but LDS only considers edge density. There are several crucial differences between LDS and LhCDS. At first, the h-clique compactness is harder to evaluate. Secondly, a h-clique spans on h vertices, making the subgraph division much more difficult. Thirdly, verification of LhCDS is more complex than verifying LDS since the clique density and clique compactness are harder to evaluate and verify.

To address the above difficulties, we jointly consider the h-clique compact number estimation and LhCDS detection, so as to design a new *iterative propose-prune-and-verify (IPPV)* pipeline. IPPV is composed of the following iterative steps: (1) estimating h-clique compact number bounds to propose LhCDS candidates; (2) pruning infeasible parts; and (3) efficient verification. To the best of our knowledge, this paper is the first to explore the LhCDS detection. The key contributions of IPPV are as follows:

(1) The initial h-clique compact number bounds are proposed based on the structures of graphs, and we prove that a convex programming, which provides h-clique diminishingly dense decomposition, can be extended to tighten the bounds.
(2) We propose a tentative graph decomposition method to deal with the case when a clique is spanning multiple subgraphs to generate correct decomposition proposals.
(3) Efficient verification is the critical part since the verification is complex for verifying both the h-clique density and h-clique compactness. We propose a novel fast verification algorithm by carefully constructing a size-reduced flow network using the maximum flow algorithm. We prove the correctness and efficiency of the proposed fast verification algorithm.
(4) At last, we further extend the *iterative propose-prune-and-verify* pipeline to detect locally general pattern densest subgraphs. More than six patterns are investigated, showing the potential of detecting locally more general pattern densest subgraphs.

We theoretically verify the exactness and efficiency of the proposed algorithm and conduct extensive experiments with different quality measures on large real datasets to verify the algorithm.

2 RELATED WORK

2.1 Densest Subgraph

The solutions to the densest subgraph (DS) problem can be classified into two categories: exact solutions and approximation solutions. The DS problem can be solved in polynomial time by exact methods based on maximum flow, linear programming, or convex optimization. Picard et al. [27] and Goldberg [13] firstly introduced the maximum-flow-based exact algorithm for the densest subgraph problem. Charikar [5] proposed an LP-based exact algorithm for the DS problem. The convex-optimization-based exact algorithm is proposed by Danisch et al. [8] and can be extended to graphs containing tens of billions of edges. Fang et al. [9] improved the efficiency of the flow-based exact algorithm by locating the densest subgraph in a specific k-core. Exact algorithms cannot scale well to large graphs, so a large number of literatures on faster approximation algorithms for the DS problem are presented.

Charikar [5] proposed a 2-approximation algorithm for the DS problem, which is known as the greedy peeling algorithm. Proving that the k_{max}-core is a 2-approximation solution to the DS problem, Fang et al. [9] improved the greedy peeling algorithm based on k_{max}-core. Inspired by the multiplicative weights update method, Boob et al. [4] designed an iterative version of the greedy peeling algorithm. Based on the MapReduce model, Bahmani et al. [2] proposed an $2(1+\epsilon)$-approximation algorithm, where $\epsilon > 0$. Based on the dual of Charikar's LP relaxation, Harb et al. [14] presented a new iterative algorithm for the DS problem. Chekuri et al. [6] proposed a flow-based approximation algorithm for the DS problem.

The DS problem has various variants focusing on different aspects and different types of graphs. Two recent surveys [18, 23] detail different variations of the DS problem and their applications to different types of graphs, such as directed graphs [5], labeled graphs [10], and uncertain graphs [40].

2.2 h-clique Densest Subgraph

Tsourakakis [35] defined the notion of h-clique density and introduced the h-clique densest subgraph (CDS) problem. Mitzenmacher et al. [25] presented a sampling scheme called the densest subgraph sparsifier, yielding a randomized algorithm that produces a well-approximate solution to the CDS problem. Fang et al. [9] proposed more efficient exact and approximation algorithms for the CDS problem. Sun et al. [33] aimed at developing near-optimal and exact algorithms for the CDS problem on large real-world graphs. They modified the Frank-Wolfe algorithm for CDS to their algorithm kClist++ and proved the effectiveness of the proposed algorithm.

2.3 Locally Densest Subgraph

The locally densest subgraph (LDS) problem is a variant of the densest subgraph (DS) problem. Qin et al. [28] proposed a method to discover the top-k representative locally densest subgraphs of a graph. The method involves defining a parameter-free definition of an LDS, showing that the set of LDSes in a graph can be computed in polynomial time, and proposing three novel pruning strategies to reduce the search space of the algorithm. Trung et al. [34] observed the hierarchical structure of maximal ρ-compact subgraphs and presented verification-free approaches to improve the efficiency of finding top-k LDSes. Ma et al. [24] proposed a convex-programming-based solution called LDScvx to the LDS problem by introducing the concept of the compact number and using the relations of compactness to the LDS problem and a specific convex program. Capitalizing on previous results [28], Samusevich et al. [31] studied the local triangle densest subgraph (LTDS) problem, which extended the LDS model to triangle based density. It's worth noting that, in essence, LDS is a specific instance of LhCDS when $h = 2$; LTDS is a specific instance of LhCDS when $h = 3$.

3 PRELIMINARIES

Given an undirected graph $G = (V, E)$, we use $\psi_h(V_{\psi_h}, E_{\psi_h})$ to denote a h-clique with $|V_{\psi_h}|$ vertices and $|E_{\psi_h}|$ edges. $\Psi_h(G)$ is the collection of h-cliques of G. $d_{\psi_h}(G)$ denotes the h-clique density of G, $d_{\psi_h}(G) = \frac{|\Psi_h(G)|}{|V|}$, and $deg_G(v, \psi_h)$ is the h-clique degree of v, i.e., the number of h-cliques containing v. Given a subset $S \subseteq V$, $G[S] = (S, E(S))$ is the subgraph induced by S, and $E(S) = E(G) \cap (S \times S)$. Table 1 summarizes the main notations used in this paper.

Table 1: MAIN NOTATIONS

Notation	Definition
$G = (V, E)$	a graph with vertex set V and edge set E
n, m	$n = \|V\|, m = \|E\|$
$G[S]$	the subgraph induced by S
$\Psi_h(G)$	the collection of h-cliques of G
$\psi_h(V_{\psi_h}, E_{\psi_h})$	a h-clique (V_{ψ_h} is vertex set, E_{ψ_h} is edge set)
$d_{\psi_h}(G)$	the h-clique density of G, $d_{\psi_h}(G) = \frac{\|\Psi_h(G)\|}{\|V\|}$
$\phi_h(u)$	h-clique compact number of vertex u
$deg_G(v, \psi_h)$	the h-clique degree of vertex v in G
$\overline{\phi}_h(u)$	the upper bounds of $\phi_h(u)$ in G
$\underline{\phi}_h(u)$	the lower bounds of $\phi_h(u)$ in G
$CP(G, h)$	the convex programming of G for h-clique densest
α	the weights distributed from h-cliques to vertices
r	the weights received by each vertex

A densest subgraph in a local region not only means that such a subgraph is not included in any other denser subgraph, but also requires the inner density to be compact and evenly distributed. Qin et al. [28] proposed the concept of ρ-compact, which gives a reasonable definition of locally densest subgraphs. A graph G is ρ-compact when removing any subset S from G removes at least $\rho \times |S|$ edges. Considering the h-clique density in a graph, we define a h-clique ρ-compact graph as:

Definition 1 (h-clique ρ-compact). *A graph $G = (V, E)$ is h-clique ρ-compact if and only if G is connected, and removing any subset of vertices $S \subseteq V$ will result in the removal of at least $\rho \times |S|$ h-cliques in G, where ρ is a non-negative real number.*

If G is h-clique ρ-compact, then h-clique degree of each vertex in G is at least $\lceil \rho \rceil$, because removing any vertex will remove at least ρ h-cliques. Besides, the h-clique density of a h-clique ρ-compact graph is at least ρ. For any $\hat{\rho} > \rho$, a h-clique $\hat{\rho}$-compact graph is also a h-clique ρ-compact graph, so we define the h-clique compactness of a graph G as the largest ρ such that G is h-clique ρ-compact. A subgraph $G[S]$ of G is a maximal h-clique ρ-compact subgraph if does not exist a supergraph of $G[S]$ is also h-clique ρ-compact.

Proposition 1. *If a graph G has h-clique density $d_{\psi_h}(G)$, then the h-clique compactness of the graph is at most $d_{\psi_h}(G)$, i.e., $\rho \le d_{\psi_h}(G)$.*

Proof. Suppose the compactness of G is evenly higher than $d_{\psi_h}(G)$, then removing all vertices in G will result in the removal of more h-cliques than $d_{\psi_h}(G) \times |V|$, which means that the h-clique density of G must be higher than $d_{\psi_h}(G)$, and is contradict to the fact that the h-clique density of G is $d_{\psi_h}(G)$. □

Proposition 1 clarifies that the h-clique compactness of a graph cannot be greater than $d_{\psi_h}(G)$. We are then interested in finding the locally h-clique dense and compact subgraph $G[S]$ in G. We formally define a locally h-clique densest subgraph as follows.

Definition 2 (Locally h-clique densest subgraph (LhCDS)). *A subgraph $G[S]$ of G is a locally h-clique densest subgraph of G if and only if $G[S]$ is a h-clique $d_{\psi_h}(G[S])$-compact subgraph, and there does not exist a supergraph $G[S']$ of $G[S]$ ($S' \supsetneq S$), such that $G[S']$ is also h-clique $d_{\psi_h}(G[S])$-compact.*

Most applications in the real world usually require finding the top-k dense regions of a graph [28], so we focus on finding the top-k LhCDSes with the largest densities. When k is large enough, all LhCDSes can be found. We formulate the problem as follows.

Definition 3 (Locally h-clique densest subgraph Problem (LhCDS Problem)). *Given a graph G, an integer h, and an integer k, the locally h-clique densest subgraph problem is to compute the top-k LhCDSes ranked by the h-clique density in G.*

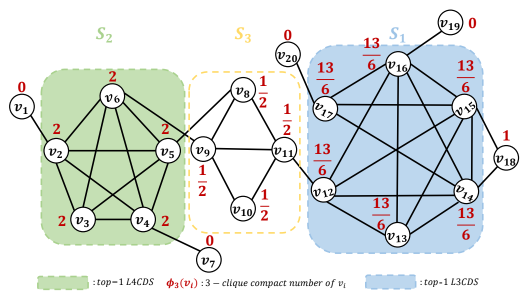


Figure 1: An example of the locally h-clique densest subgraph

Figure 1 shows an example of the LhCDS. We use S_1,S_2, and S_3 to represent $\{v_{12}, ..., v_{17}\}$, $\{v_2, ..., v_6\}$, and $\{v_8, ..., v_{11}\}$. When $h = 3$, the top-1 L3CDS is $G[S_1]$, which has a 3-clique density of $\frac{13}{6}$, since there are thirteen 3-cliques in it. The top-1 and top-2 L4CDSes are $G[S_2]$ and $G[S_1]$. They both have a 4-clique density of 1.

Note that an edge in a graph G is a 2-clique; therefore, the intensively studied LDS problem [24, 28] can be seen as an instance of the LhCDS problem when $h = 2$. Similarly, the LTDS problem [31] is exactly the L3CDS problem. Therefore, the LhCDS problem studied in this paper provides a more general framework, and we boldly infer that our method can be generalized from h-clique to general patterns, which means that we can give an algorithmic framework to solve a wider range of locally pattern densest problems.

4 LhCDS DISCOVERY

In this section, we focus on the design of the LhCDS discovery problem. According to the concept of h-clique compactness of a graph G, each subgraph of a graph G has its own compactness. However, each vertex may be contained in various subgraphs with different compactness. Therefore, we introduce the concept of h-clique compact number for each vertex in a graph.

Definition 4 (h-clique compact number). *Given a graph $G = (V, E)$, for each vertex $u \in V$, the h-clique compact number of u is the largest ρ such that u is contained in a h-clique ρ-compact subgraph of G, denoted by $\phi_h(u)$.*

In the following theorem, we prove the relationship between the LhCDS and the h-clique compact numbers of vertices within it.

Theorem 1. *Given an LhCDS $G[S]$ in G, for each vertex $u \in S$, the h-clique compact number of u equals to the h-clique density of $G[S]$, i.e., $\phi_h(u) = d_{\psi_h}(G[S])$.*

Proof. As $G[S]$ is a maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph, for each $u \in S$, there exists no other subgraph $G[S']$ containing u such that $G[S']$ is a h-clique ρ-compact subgraph with $\rho > d_{\psi_h}(G[S])$. We prove the claim by contradiction. Suppose $G[S']$ is a h-clique ρ-compact subgraph with $\rho > d_{\psi_h}(G[S])$ and $u \in S'$, we have $d_{\psi_h}(G[S']) \geq \rho > d_{\psi_h}(G[S])$. First $S' \subseteq S$, because $G[S]$ is a maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph and $S' \cap S \neq \emptyset$. If we remove $U = S \backslash S'$ from $G[S]$, the number of h-cliques removed is $|\Psi_h(G[S])| - |\Psi_h(G[S'])| = d_{\psi_h}(G[S]) \times |S| - d_{\psi_h}(G[S']) \times |S'| < d_{\psi_h}(G[S]) \times (|S| - |S'|) = d_{\psi_h}(G[S]) \times |U|$. This contradicts that $G[S]$ is h-clique $d_{\psi_h}(G[S])$-compact. Hence, $d_{\psi_h}(G[S])$ is the h-clique compact number of all vertices in S. □

Based on Theorem 1, once we get the h-clique compact number of each vertex in G, we can obtain top-k LhCDSes. For example, in Figure 1, we list the 3-clique compact numbers of all vertices of G. It is obvious that $G[S_1]$ and $G[S_2]$ are both L3CDSes.

However, computing the h-clique compact numbers directly and accurately is difficult. So we jointly consider h-clique compact number and LhCDS to design a new "iterative propose-prune-and-verify" pipeline for top-k LhCDS detection, which is called IPPV. In the proposal part, the true LhCDSes are allowed to be encapsulated in the proposed candidates, but without missing true LhCDSes. Proper graph decomposition method should be designed, since a clique may span multiple subgraphs to be decomposed. In the verification part, each correct LhCDS should be outputted, and LhCDS candidates that can be further pruned should be indicated.

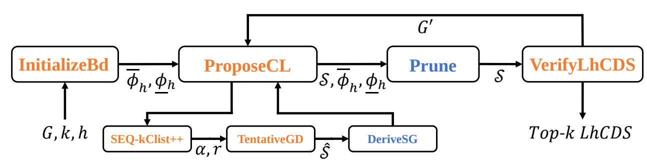


Figure 2: Flow diagram of IPPV

Figure 2 gives the flow diagram of IPPV. It has four main parts: **1)** calculating the initial bounds of the h-clique compact numbers of vertices; **2)** iteratively proposing all LhCDS candidates (generating approximate h-clique compact numbers; decomposing the graph tentatively; grouping vertices and tightening bounds); **3)** pruning invalid vertices; **4)** verifying the locally densest property of all candidates to find top-k LhCDS, and we, in particular, propose a basic algorithm and a fast algorithm for verification. As a general algorithm framework, all blue parts are extensions of existing methods, and all orange parts are our proof and innovation for this problem.

4.1 Initial h-clique Compact Number Bounds

In order to derive LhCDS candidates, we first give initial upper and lower bounds of h-clique compact number $\phi_h(u)$. Specifically, we denote $\overline{\phi}_h(u)$ and $\underline{\phi}_h(u)$ as the upper and lower bound of $\phi_h(u)$ in

G. We use (k, ψ_h)-core[9], which is a cohesive subgraph model to compute the initial bounds.

Definition 5 (**(k, ψ_h)-core).** *Given a graph G, the (k, ψ_h)-core is the largest subgraph of G, in which the h-clique degree of each vertex is at least k. The h-clique-core number of a vertex $u \in V$, denoted by $core_G(u, \psi_h)$, is the highest k of (k, ψ_h)-core containing u.*

Proposition 2. *h-clique compact number $\phi_h(u)$ has following relations to the h-clique-core number $core_G(u, \psi_h)$.*

(1) A (k, ψ_h)-core subgraph is h-clique $\frac{k}{h}$-compact. For any $u \in V$, $\underline{\phi}_h(u)$ can be assigned as $\frac{core_G(u,\psi_h)}{h}$;

(2) If $G[S]$ is an LhCDS of G, for all $u \in S$, $core_G(u, \psi_h) \geq d_{\psi_h}(G[S])$. For any $u \in V$, $\overline{\phi}_h(u)$ can be assigned as $core_G(u, \psi_h)$.

Proof. Any vertex in a (k, ψ_h)-core subgraph is contained in at least k h-cliques. By removal of any subset S from the (k, ψ_h)-core, at least $\frac{k}{h} \times |S|$ h-cliques would be removed. For any $u \in V$, there is a h-clique $\frac{core_G(u,\psi_h)}{h}$-compact subgraph of G that contains u, then $\frac{core_G(u,\psi_h)}{h}$ is an lower bound of $\phi_h(u)$. The second relation can be obtained from the fact that an LhCDS $G[S]$ in a graph G is a $(\lceil d_{\psi_h}(G[S]) \rceil, \psi_h)$-core subgraph of G. For any $u \in V$, if an LhCDS containing u, then $core_G(u, \psi_h)$ is an upper bound of $\phi_h(u)$. □

Algorithm 1: The bound initialization algorithm

Input: $G = (V, E), h$
Output: $\overline{\phi}_h, \underline{\phi}_h$
1 **foreach** $u \in V$ **do** compute $core_G(u, \psi_h)$;
2 **foreach** $u \in V$ **do**
3 $\quad \overline{\phi}_h(u) \leftarrow core_G(u, \psi_h)$; $\underline{\phi}_h(u) \leftarrow \frac{core_G(u,\psi_h)}{h}$;
4 **return** $\overline{\phi}_h, \underline{\phi}_h$;

According to Proposition 2, we can get the initial bounds of h-clique compact number $\phi_h(u)$ of G (Lines 2-3) by Algorithm 1.

4.2 Candidate LhCDS Proposal

The initial upper and lower bounds for h-clique compact numbers from h-clique-core numbers are relatively loose. In this section, we focus on how to tighten the bounds and propose LhCDS candidates.

4.2.1 Overall Algorithm for Candidate LhCDS Proposal. The overall candidate LhCDS proposal algorithm is given in Algorithm 2. Approximate h-clique compact number is calculated via `SEQ-kClist++` (Line 1); the preliminary partition of G and recalculated values are obtained via `TentativeGD` (Line 2); tighter upper and lower bounds for h-clique compact numbers and the further partition of G (stable h-clique group) are calculated via `DeriveSG` (Line 3). The sub-procedures introduce each of the above functions (Lines 5-33).

4.2.2 Generate Approximate h-clique Compact Number. Inspired from a classical convex programming [8, 24], we propose a convex programming for finding the diminishingly-h-clique-dense decomposition, and prove that the optimal solution of our convex programming is exactly the h-clique compact number of a graph G.

Algorithm 2: The candidate LhCDS proposal algorithm

Input: $G = (V, E)$, number of iterations T, $\overline{\phi}_h, \underline{\phi}_h$
Output: $\mathcal{S}, \overline{\phi}_h, \underline{\phi}_h$
1 $(\alpha, r) \leftarrow$ `SEQ-kClist++` (G', T) ;
2 $\hat{\mathcal{S}}, \alpha, r \leftarrow$ `TentativeGD` (G, α, r);
3 $\mathcal{S}, \overline{\phi}_h, \underline{\phi}_h \leftarrow$ `DeriveSG` $(\hat{\mathcal{S}}, \alpha, r, \overline{\phi}_h, \underline{\phi}_h)$;
4 return $\mathcal{S}, \overline{\phi}_h, \underline{\phi}_h$;
5 **Procedure** `SEQ-kClist++`(G', T)
6 $\quad$ **foreach** *h-clique ψ_h in G* **do** $\alpha_{u,\psi_h} \leftarrow \frac{1}{h}$, $\forall u \in V_{\psi_h}$;
7 $\quad$ **foreach** $u \in V$ **do** $r(u) \leftarrow \sum_{\psi_h \in \Psi_h(G): u \in \psi_h} \alpha_{u,\psi_h}$;
8 $\quad$ **foreach** *iteration t=1,...,T* **do**
9 $\quad\quad \gamma_t \leftarrow \frac{1}{t+1}$; $\alpha \leftarrow (1 - \gamma_t) * \alpha$; $r \leftarrow (1 - \gamma_t) * r$;
10 $\quad\quad$ **foreach** *h-clique ψ_h* **do**
11 $\quad\quad\quad v_{min} \leftarrow argmin_{v \in \psi_h} r(v)$;
12 $\quad\quad\quad \alpha_{v_{min},\psi_h} \leftarrow \alpha_{v_{min},\psi_h} + \gamma_t$; $r(v_{min}) \leftarrow r(v_{min}) + \gamma_t$;
13 $\quad$ **return** (α, r);
14 **Procedure** `TentativeGD`(G, α, r)
15 $\quad$ sort vertices in V in descending order according to r;
16 $\quad P \leftarrow \{p | p = argmax_{p \leq q \leq n} d_{\psi_h}(G[V_{[1:q]}])\}$;
17 $\quad \hat{\mathcal{S}} \leftarrow$ partition V according to P;
18 $\quad$ **foreach** $\psi_h \in \Psi_h(G)$ **do**
19 $\quad\quad p \leftarrow \max\{1 \leq i \leq l : \psi_h \cap \hat{S}_i \neq \emptyset\}$;
20 $\quad\quad s \leftarrow \sum_{u \in \psi_h \setminus \hat{S}_p} \alpha_{u,\psi_h}$;
21 $\quad\quad \forall u \in \psi_h \setminus \hat{S}_p, \alpha_{u,\psi_h} \leftarrow 0$;
22 $\quad\quad \forall u \in \psi_h \cap \hat{S}_p, \alpha_{u,\psi_h} \leftarrow \alpha_{u,\psi_h} + \frac{s}{|\psi_h \cap \hat{S}_p|}$;
23 $\quad$ **foreach** $u \in V$ **do** $r(u) \leftarrow \sum_{\psi_h \in \Psi_h(G): u \in \psi_h} \alpha_{u,\psi_h}$;
24 $\quad$ **return** $\hat{\mathcal{S}}, \alpha, r$;
25 **Procedure** `DeriveSG`$(\hat{\mathcal{S}}, \alpha, r, \overline{\phi}_h, \underline{\phi}_h)$
26 $\quad$ **while** *$\hat{\mathcal{S}}$ is not empty* **do**
27 $\quad\quad S' \leftarrow$ pop out the first candidate from $\hat{\mathcal{S}}$; $S \leftarrow S \cup S'$;
28 $\quad\quad$ **if** *S is a stable h-clique group* **then** put S into $\mathcal{S}$; $S \leftarrow \emptyset$;
29 $\quad$ **foreach** $S \in \mathcal{S}$ **do**
30 $\quad\quad$ **foreach** $u \in S$ **do**
31 $\quad\quad\quad \overline{\phi}_h(u) \leftarrow min\{\overline{\phi}_h(u), max_{v \in S} r(v)\}$;
32 $\quad\quad\quad \underline{\phi}_h(u) \leftarrow max\{\underline{\phi}_h(u), min_{v \in S} r(v)\}$;
33 $\quad$ **return** $\mathcal{S}, \overline{\phi}_h, \underline{\phi}_h$;

Intuitively, the aim of CP(G, h) is that each h-clique $\psi_h \in \Psi_h(G)$ tries to distribute its unit weight among its h vertices such that the sum of the weight received by the vertices are as even as possible. We use α_{u,ψ_h} to represent the weight assigned to u from h-clique ψ_h and $r(u)$ to denote the sum of the weight assigned to u from h-cliques that containing u. This intuition suggests that we can consider the objective function: $Q_{G,h}(\alpha) := \sum_{u \in V} r(u)^2$, in which $r(u) = \sum_{\psi_h \in \Psi_h(G): u \in \psi_h} \alpha_{u,\psi_h}$, for all $u \in V$. The convex programming is:

$$\mathrm{CP}(G, h) := \min \left\{ Q_{G,h}(\alpha) : \alpha \in \mathcal{D}(G, h) \right\},$$

where the domain is:

$$\mathcal{D}(G, h) := \left\{ \alpha \in \prod_{\psi_h \in \Psi_h(G)} \mathbb{R}_+^{\psi_h} : \forall \psi_h \in \Psi_h(G), \sum_{u \in \psi_h} \alpha_{u,\psi_h} = 1 \right\}.$$

Here, we demonstrate that the h-clique compact numbers can be derived from the optimal solution of CP(G, h).

Theorem 2. *Suppose (α^*, r^*) is an optimal solution of CP(G, h). Then, $\forall u \in V$, $\phi_h(u) = r^*(u)$, i.e., each $r^*(u)$ in r^* is exactly the h-clique compact number of u.*

Proof. For any vertex $u \in V$, let $S^+ = \{v \in V | r^*(v) > r^*(u)\}$, $S^= = \{v \in V | r^*(v) = r^*(u)\}$, $S^- = \{v \in V | r^*(v) < r^*(u)\}$, $u \in S^=$. $S^{+=}$ denotes the vertices that are contained by h-cliques that including vertices both in S^+ and $S^=$. We prove $G[S^+ \cup S^=]$ is a h-clique $r^*(u)$-compact subgraph. First, removing $S^=$ from $G[S^+ \cup S^=]$ will result in the removal of $r^*(u) \times |S^=|$ cliques in $G[S^+ \cup S^=]$. We know that for all $(v, w) \in E \cap (S^+ \times S^=)$, $r^*(v) > r^*(w)$ and $\alpha_{v,\psi_h(v,w \in \psi_h)} = 0$. Otherwise, if there exists $(v, w) \in E \cap (S^+ \times S^=)$ such that $\alpha_{v,\psi_h(v,w \in \psi_h)} > 0$, there exists $r^*(v) - r^*(w) > \epsilon > 0$. We can reduce $\alpha_{v,\psi_h(v,w \in \psi_h)}$ by ϵ and increase $\alpha_{w,\psi_h(v,w \in \psi_h)}$ by ϵ, and the objective function be decreased by $2\epsilon(r^*(v) - r^*(w) - \epsilon)$, which contradicts the optimality of r^*. Similarly, we can prove that for all $(v, w) \in E \cap (S^= \times S^-)$, $r^*(v) > r^*(w)$ and $\alpha_{v,\psi_h(v,w \in \psi_h)} = 0$. Therefore, $r^*(u) \times |S^=| = \sum_{\psi_h \in \Psi_h(G): v \in S^=, v \in \psi_h} \alpha_{v,\psi_h} = |\Psi_h(G[S^=]) \cup \Psi_h(G[S^{+=}])|$. $r^*(u) \times |S^=|$ is exactly the number of h-cliques to be removed when removing $S^=$ from $G[S^+ \cup S^=]$. Meanwhile, for any $S' \subseteq S^+ \cup S^=$, we have that $r^*(u) \times |S'| \leq \sum_{\psi_h \in \Psi_h(G): v \in S', v \in \psi_h} \alpha_{v,\psi_h} \leq \sum_{\psi_h \in \Psi_h(G[S^+ \cup S^=]): v \in S', v \in \psi_h} 1$, which means removing any $S' \subseteq S^+ \cup S^=$ from $G[S^+ \cup S^=]$ will result in the removal of at least $r^*(u) \times |S'|$ h-cliques. Therefore, $G[S^+ \cup S^=]$ is a h-clique $r^*(u)$-compact subgraph. Analogously, we can prove that for any other subset S'' containing u, $G[S'']$ is a h-clique ρ-compact subgraph, where $\rho \leq r^*(u)$, by contradiction. Therefore, $r^*(u)$ is the largest ρ such that u is contained in a h-clique ρ-compact subgraph of G, which is exactly the h-clique compact number of u. □

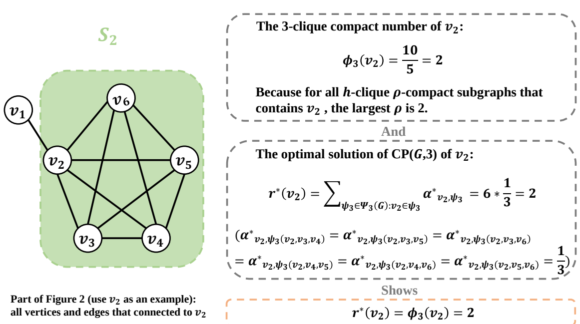


Figure 3: An example of the relationship between $r^*(u)$ and $\phi_h(u)$ of a vertex u in G

Consider the convex programming CP(G, 3) for G in Figure 1, we use v_2 as an example, shown in Figure 3. The 3-clique compact number of v_2 is 2, and the optimal solution $r^*(v_2)$ value is also 2. It is clear that for each $u \in V$, $r^*(u)$ is exactly $\phi_h(u)$.

Exactly attaining the (α^*, r^*) is difficult, so we use the approximate solution (α, r) of CP(G, h) to tighten the h-clique compact bounds. Frank-Wolfe-based algorithm is efficient for finding approximate solutions of CP(G) [24]. However, FW-based algorithm for h-clique densest requires a large amount of memory. `SEQ-kClist++`[33] is better for approximately calculating α_{u,ψ_h} for each h-clique ψ_h, $u \in \psi_h$, as well as $r(u)$ for each vertex u. All α_{u,ψ_h} are initialized to $\frac{1}{h}$ (Line 6). $r(u)$ stores the sum over all α_{u,ψ_h}'s such that ψ_h contains u (Line 7). At each iteration, α and r are modified simultaneously as follows. For each h-clique ψ_h, we find the minimum $r(v_{min})$ among ψ_h, and new values for the α_{v_{min},ψ_h} and the $r(v_{min})$ are computed as convex combinations (Lines 8-12).

4.2.3 Tentative Graph Decomposition. After getting approximate (α, r), we can derive a graph decomposition from the given (α, r).

Proposition 3. *Given an LhCDS $G[S]$ in G, $\forall (u, v) \in E$, if $u \in S$ and $v \in V \backslash S$, we have $\phi_h(u) > \phi_h(v)$.*

Considering vertices adjacent to $G[S_1]$ but not in $G[S_1]$ in Figure 1, such as v_{11} and v_{18}, their 3-clique compact numbers fulfill Proposition 3: $\phi_3(v_{11}) = \frac{1}{2} < \frac{13}{6}$, $\phi_3(v_{18}) = 1 < \frac{13}{6}$.Proposition 3 is helpful for choosing LhCDSes from all subgraphs.

Proposition 4 (Disjoint property). *Suppose $G[S]$ and $G[S']$ are two LhCDSes in G, we have $S \cap S' = \emptyset$.*

Proof. Without loss of generality, we suppose $d_{\psi_h}(G[S]) \geq d_{\psi_h}(G[S'])$. We prove the proposition by contradiction. Suppose $S \cap S' \neq \emptyset$. According to the definition of LhCDS, $G[S] \nsubseteq G[S']$. Since $G[S]$ and $G[S']$ are two LhCDSes, the graph induced by $S \bigcup S'$ is a connected h-clique $d_{\psi_h}(G[S'])$- compact graph which is larger than S'. That contradicts the fact that $G[S']$ is an LhCDS. □

Proposition 4 proves that all the LhCDSes in a graph G are pairwise disjoint. Therefore, the number of LhCDSes of G is bounded by $|V|$, and the LhCDSes can be used to identify all the non-overlapping h-clique dense regions of a graph.

We then propose `TentativeGD` to generate tentative graph decomposition for proposing LhCDS. The vertices in V are sorted based on r values descendingly (Line 15). The initial partition $\hat{\mathcal{S}}$ of the graph is extracted based on the descending order (Lines 16-17). For each $\psi_h \in \Psi_h(G)$, if the clique ψ_h is contained in multiple vertex sets, the vertex set with the largest set index will be recorded as p, and the α value of ψ_h will be redistributed to vertices in $\hat{S}_p$ (Lines 18-22). In other words, for the convenience of partition, the α value of ψ_h straddling multiple vertex sets is redistributed to a vertex set with the lowest r value. Finally, the r values of all vertices in V are recalculated (Line 23).

4.2.4 Stable h-clique Group Derivation. After getting the initial bounds of h-clique compact numbers in `InitializeBd` and a preliminary partition of the graph in `TentativeGD`, we consider obtaining the tighter bounds of h-clique compact numbers and a further partition of the graph, to calculate LhCDS candidates. Inspired by two concepts, stable subset [8] and stable group [24], for solving the h-clique densest subgraph problem, we propose the definition of the stable h-clique group.

Definition 6 (stable h-clique group). *Given a feasible solution (α, r) to CP(G, h), a stable h-clique group with respect to (α, r) is a non-empty vertex group $S \in V$, if the following conditions hold.*

(1) For any $v \in V \backslash S$, $r(v) > max_{u \in S} r(u)$ or $r(v) < min_{u \in S} r(u)$;
(2) For any $v \in V$, if $r(v) > max_{u \in S} r(u)$, $\forall \psi_h (u, v \in \psi_h)$, $\alpha_{v,\psi_h} = 0$;
(3) For any $v \in V$, if $r(v) < min_{u \in S} r(u)$, $\forall \psi_h (u, v \in \psi_h)$, $\alpha_{u,\psi_h} = 0$.

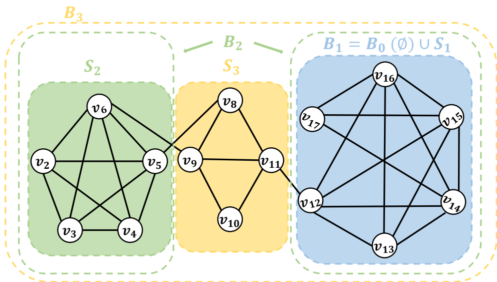


Figure 4: The relations between stable 3-clique subset $\mathcal{B}$ and stable 3-clique group $\mathcal{S}$

The concept of the stable h-clique subset $\mathcal{B}$ is related to the stable h-clique group $\mathcal{S}$, and the relations between stable h-clique subset and stable h-clique group can be shown in Figure 4 with $h = 3$. All stable h-clique groups are disjoint, and a stable h-clique subset is the union of the previous stable h-clique subset and the first stable h-clique group outside this previous stable h-clique subset. Either $\mathcal{B}$ or $\mathcal{S}$ can form a consecutive subsequence of the whole sequence, and we only use the stable h-clique group in our algorithm.

Theorem 3. *Given a feasible solution (α, r) to CP(G, h) and a stable h-clique group S with respect to (α, r), for all $v \in S$, we have that $min_{u \in S} r(u) \le \phi_h(v) \le max_{u \in S} r(u)$.*

Proof. According to Theorem 2, for all $u \in V, r^*(u) = \phi_h(u)$. Suppose there exists a vertex $v \in S$ such that $r^*(v) = \phi_h(v) < min_{u \in S} r(u) \le r(v)$. Since $\sum_{u \in V} r(u) = \sum_{u \in V} r^*(u)$, correspondingly, there must exist another vertex $w \in V$, $r^*(w) = \phi_h(w) > r(w)$. The difference between $r(w)$ and $r^*(w)$ means that there exists ψ_h contains both v and w, $\alpha_{v,\psi_h} > 0$. Since S is a stable h-clique group, according to the third condition in Definition 6, $r(w) > min_{u \in S} r(u)$. There exists $\epsilon > 0$, we can increase $r^*(v)$ by ϵ and decrease $r^*(w)$ by ϵ to decrease the value of the objective function. This contradicts that r^* is the optimal solution to CP(G, h). By the same token, for all $u \in V, r^*(u) = \phi_h(u) \le max_{u \in S} r(u)$. □

Based on Theorem 3, the stable h-clique groups can give tighter bounds of h-clique compact numbers, so we propose `DeriveSG` algorithm to derive the stable h-clique groups, which are our LhCDS candidates. In `DeriveSG`, the subsets in $\hat{\mathcal{S}}$ are checked one by one; if the subset is a stable h-clique group, it will be pushed into the set of stable h-clique groups $\mathcal{S}$; otherwise, in the next iteration, the current subset S will be merged with the next subset S' (Lines 26–28). Then, the upper and lower bounds of h-clique compact numbers are updated based on Theorem 3 (Lines 29–32).

4.3 Pruning for Candidate LhCDS Derivation

We prove that the following lemma can help to prune invalid vertices that are certainly not contained by any LhCDS.

Lemma 1. *For any $v \in V$, v is not contained by any LhCDS in G if either of the following two conditions is satisfied.*

(1) If there exists $(u, v) \in E$, such that $\underline{\phi}_h(u) > \overline{\phi}_h(v)$, v is invalid;

(2) Let G' denote the graph after pruning all invalid vertices in condition (1). $\overline{\phi}_h^G(u)$ is the upper bound of $\phi_h^G(u)$ in G. For any u in G', if $\overline{\phi}_h^{G'}(v) < \underline{\phi}_h(v)$, v is invalid.

Proof. First,we prove condition (1). For any $u, v \in V$, $(u, v) \in E$, if $\underline{\phi}_h(u) > \overline{\phi}_h(v)$, then $\phi_h(u) > \phi_h(v)$. According to Proposition 3, v is not contained in LhCDS,i.e., v is invalid.

For condition (2), $\overline{\phi}_h^{G'}(u) < \underline{\phi}_h(u)$ means that to form a h-clique $\underline{\phi}_h(u)$-compact subgraph containing u, some already pruned vertices are needed. So using the vertices in G' only cannot form a h-clique $\underline{\phi}_h(u)$-compact subgraph containing u. Therefore, u cannot be contained by any LhCDS in G,i.e., v is invalid. □

According to Lemma 1, we design Pruning Rule to prune invalid vertices by condition (1) and condition (2). An example can be seen in Figure 1 with $h = 3$. v_9 and v_{11} can be pruned, because for edge (v_6, v_9), $\underline{\phi}_3(v_6) = 2 > \overline{\phi}_3(v_9) = \frac{1}{2}$; for edge (v_{11}, v_{12}), $\underline{\phi}_3(v_{12}) = \frac{13}{6} > \overline{\phi}_3(v_{11}) = \frac{1}{2}$. Analogously, the vertices v_1, v_7, v_{18}, v_{19}, and v_{20} are also pruned by condition (1).

We denote the graph after pruning by G'. Some vertices in G' become invalid vertices, because any LhCDS in G containing these vertices needs to include some already pruned vertices, which can not form an LhCDS in G'. Therefore, we utilize condition (2). Based on Proposition 2, for any vertex $u \in V(G')$, $core_G(u, \psi_h)$ provides an upper bound of $\phi_h^{G'}(u)$. For example, after v_9 and v_{11} are pruned, the upper bounds of 3-clique compact numbers of v_8 and v_{10} in graph G' are $\overline{\phi}_3^{G'}(v_8) = \overline{\phi}_3^{G'}(v_{10}) = 0 < \underline{\phi}_3(v_8) = \underline{\phi}_3(v_{10}) = \frac{1}{2}$. So v_8 and v_{10} are pruned using condition (2).

Based on Lemma 1, we propose `Prune` algorithm shown in Algorithm 3. G is replicated to G' for pruning (Line 1). Condition (1) is used to remove invalid vertices in G' (Lines 2-3); after computing the h-clique core numbers for all vertices in G' (Line 4), condition (2) is applied to further remove invalid vertices in G' (Lines 5-7). Finally, the LhCDS candidates are updated from the intersection of h-clique stable groups and the unpruned vertex sets (Line 8).

Algorithm 3: The pruning algorithm

Input: $G = (V, E)$, $\mathcal{S}, \overline{\phi}_h, \underline{\phi}_h$
Output: $\mathcal{S}$
1 $G' = (V(G'), E(G')) \leftarrow G$;
2 **foreach** $(u, v) \in E$ **do**
3   **if** $\overline{\phi}_h(v) < \underline{\phi}_h(u)$ **then** remove v from G' ;
4 **foreach** $u \in V(G')$ **do** compute $core_{G'}(u, \psi_h)$;
5 **while** *there exists* $u \in V(G'), core_{G'}(u, \psi_h) < \underline{\phi}_h(u)$ **do**
6   remove u from G';
7   update h-clique-core numbers of vertices adjacent to u;
8 **foreach** *LhCDS candidate* $S \in \mathcal{S}$ **do** $S \leftarrow S \cap V(G')$;
9 return $\mathcal{S}$;

4.4 LhCDS Verification

Since candidate LhCDSes are obtained approximately, we need to confirm whether the candidates are LhCDSes.

Proposition 5. *There are the following properties about an LhCDS.*

(1) any subgraph of an LhCDS cannot be denser than itself;
(2) An LhCDS itself is compact, and any supergraph of an LhCDS cannot be more compact than itself.

Proof. (2) of Proposition 5 can be directly obtained from the definition of LhCDS. We prove (1) by contradiction. Suppose that there is a subgraph $G[S']$ in an LhCDS $G[S]$, $S' \subset S$, such that $d_{\psi_h}(G[S']) > d_{\psi_h}(G[S])$. By removal of the set $U = S \backslash S'$from $G[S]$, we remove $|\Psi_h(G[S])| - |\Psi_h(G[S'])|$ h-cliques. Note that $|\Psi_h(G[S])| - |\Psi_h(G[S'])| = d_{\psi_h}(G[S])|S| - d_{\psi_h}(G[S'])|S'| < d_{\psi_h}(G[S])(|S| - |S'|) = d_{\psi_h}(G[S])|U|$, which contradicts the fact that $G[S]$ is an LhCDS, i.e. h-clique $d_{\psi_h}(G[S])$-compact. □

We need to verify: **1)** whether a candidate LhCDS $G[S]$ is self-densest and **2)** whether $G[S]$ is a maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph in G. We use `IsDensest` [33] algorithm to check whether a candidate LhCDS $G[S]$ is self-densest. In this section, we focus on the verification of the second property, to verify whether $G[S]$ is a connected component of maximal h-clique $d_{\psi_h}(G[S])$-compact subgraphs in G. We design a basic verification algorithm, and to reduce the scale of the flow network, we further propose a fast algorithm. The correctness of both algorithms is proved.

4.4.1 Basic Verification Algorithm. Given an LhCDS candidate $G[S]$, we propose a innovative flow network to derive maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph G' in G. If $G[S]$ is a connected component of G', $G[S]$ is indeed maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph and an LhCDS in G; otherwise, $G[S]$ is not an LhCDS. The flow network $\mathcal{F}(V_{\mathcal{F}}, E_{\mathcal{F}})$ is shown in Figure 5. The vertex set of $\mathcal{F}$ is $\{s\} \cup V \cup \Psi_h \cup \{t\}$. The arc set of $\mathcal{F}$ is given as follows. For each h-clique ψ_h^j, we add h incoming arcs of capacity 1 from the vertices which form ψ_h^j, and h outgoing arcs of capacity of $h-1$ to the same set of vertices. For each vertex $v_i \in V$, we add an incoming arc of capacity $deg_G(v_i, \psi_h)$ from the source vertex s, and an outgoing arc of capacity $\rho * h$ to the sink vertex t. Given a parameter ρ, we prove that the flow network in `DeriveCompact` can be used to derive maximal h-clique ρ-compact subgraphs in G according to Theorem 4.

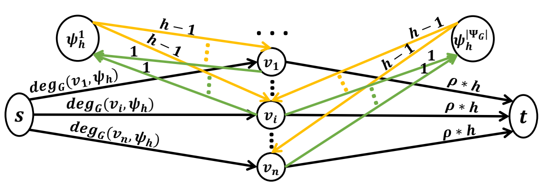


Figure 5: The flow network of `DeriveCompact`$(G, \rho, \emptyset)$

Theorem 4. *If G contains maximal h-clique ρ-compact subgraphs, then the result returned by DeriveCompact $(G, \rho - \frac{1}{|V|^2}, \emptyset)$ is the set of all maximal h-clique ρ-compact subgraphs in G.*

Proof. Based on Proposition 4, two LhCDSes are disjoint. We use $G[S_1]$ to represent the union of all maximal h-clique ρ-compact subgraphs in G. $G[S_2]$ denotes the subgraph returned by DeriveCompact $(G, \rho - \frac{1}{|V|^2}, \emptyset)$, which is the largest subgraph in G with maximum $|\Psi_h(G[S_2])| - \rho \times |S_2|$ [13][9]. We prove that $G[S_1]$ and $G[S_2]$ are the same. First, we prove that $G[S_2]$ is a subgraph of $G[S_1]$ by contradiction. Suppose a connected component $G[S]$ of $G[S_2]$ is not h-clique ρ-compact, then there exists a subset $S' \subseteq S$ such that removing S' from S will result in removing less h-cliques than $\rho \times |S'|$, then $|\Psi_h(G[S])| - |\Psi_h(G[S \backslash S'])| < \rho \times |S'| = \rho \times (|S| - |S \backslash S'|)$. We have $|\Psi_h(G[S])| - \rho \times |S| < |\Psi_h(G[S \backslash S'])| - \rho \times |S \backslash S'|$. Therefore, replacing $G[S]$ by its subgraph $G[S \backslash S']$ in $G[S_2]$ will enlarge the value of $|\Psi_h(G[S_2])| - \rho \times |S_2|$, which contradicts the condition that $G[S_2]$ has the maximum $|\Psi_h(G[S_2])| - \rho \times |S_2|$. Second, we prove that $G[S_1]$ is a subgraph of $G[S_2]$ by contradiction. Suppose $G[S_1]$ is not a subgraph of $G[S_2]$, according to the result before, we have $S_2 \subset S_1$. There exists a subset $S \neq \emptyset$ and $S = S_1 \backslash S_2$. Removing S from $G[S_1]$ will result in removing at least $\rho \times |S|$ h-cliques, then $|\Psi_h(G[S_1])| - |\Psi_h(G[S_2])| \geq \rho \times |S| = \rho \times (|S_1| - |S_2|)$. We have $|\Psi_h(G[S_1])| - \rho \times |S_1| \geq |\Psi_h(G[S_2])| - \rho \times |S_2|$, so enlarging $G[S_2]$ to $G[S_1]$ will not decrease the value of $|\Psi_h(G[S_2])| - \rho \times |S_2|$, which contradicts the condition that $G[S_2]$ has the maximum $|\Psi_h(G[S_2])| - \rho \times |S_2|$. Therefore, the theorem is proved. □

Algorithm 4: The basic LhCDS verification algorithm

Input: $G(V, E), S$
Output: VerifyLhCDS
1 $\rho \leftarrow d_{\psi_h}(G[S])$, VerifyL$h$CDS $\leftarrow$ True;
2 $G' \leftarrow$ `DeriveCompact` $(G, \rho - \frac{1}{|V|^2}, \emptyset)$;
3 **return** $G[S]$ is a connected component in G';
4 **Procedure** `DeriveCompact`(G, ρ, P)
5 $\quad$ $cnt \leftarrow 0$; $\Psi_h \leftarrow$ all the instances of h-clique ψ_h in G;
6 $\quad$ $V_{\mathcal{F}} \leftarrow \{s\} \cup V \cup \Psi_h \cup P \cup \{t\}$;
7 $\quad$ **foreach** $\psi_h \in \Psi_h$ **do**
8 $\quad\quad$ **foreach** $v \in \psi_h$ **do**
9 $\quad\quad\quad$ add an edge $\psi_h \rightarrow v$ with capacity $h-1$;
10 $\quad\quad\quad$ add an edge $v \rightarrow \psi_h$ with capacity 1;
11 $\quad$ **foreach** $\psi_h \in P$ **do**
12 $\quad\quad$ $cnt \leftarrow cnt$ of ψ_h;
13 $\quad\quad$ **foreach** $v \in \psi_h$ *and* $v \in G$ **do**
14 $\quad\quad\quad$ add an edge $\psi_h \rightarrow v$ with capacity $h-1$;
15 $\quad\quad\quad$ add an edge $v \rightarrow \psi_h$ with capacity $1 + \frac{h-cnt}{cnt}$;
16 $\quad\quad\quad$ $deg_G(v, \psi_h) \leftarrow deg_G(v, \psi_h) + 1 + \frac{h-cnt}{cnt}$;
17 $\quad$ **foreach** $v \in V$ **do**
18 $\quad\quad$ add an edge $v \rightarrow t$ with capacity $\rho * h$;
19 $\quad\quad$ add an edge $s \rightarrow v$ with capacity $deg_G(v, \psi_h)$;
20 $\quad$ Compute the minimum $s-t$ cut $(\mathcal{S}, \mathcal{T})$ from the flow network $\mathcal{F}(V_{\mathcal{F}}, E_{\mathcal{F}})$;
21 $\quad$ **return** $G[\mathcal{S} \setminus s]$;

In Algorithm 4, we first derive all connected components of the h-clique $d_{\psi_h}(G[S])$-compact subgraph G' in G by `DeriveCompact` (Line 2). If $G[S]$ is a connected component of G', the algorithm return True (Line 3). In `DeriveCompact`, all the instances of h-clique is collected (Line 5). To build a flow network $\mathcal{F}(V_{\mathcal{F}}, E_{\mathcal{F}})$, a vertex set $V_{\mathcal{F}}$ is created, and vertices in $V_{\mathcal{F}}$ are linked by directed edges with different capacities (Lines 6-19). Then, the minimum $s-t$ cut $(\mathcal{S}, \mathcal{T})$ is computed (Line 20).

4.4.2 Fast Verification Algorithm. Although the basic verification algorithm can successfully verify whether a given subset is LhCDS, the scale of the flow network in algorithm 4 is large, and the running time is long in large-scale graphs. We prove that the verification can be done by verifying only the subgraph $G[S]$ and the vertices around the subgraph $G[S]$, which is denoted by $G[T]$. Since $G[T]$ is much smaller than G, checking the minimum cut in $G[T]$ is much more efficient. Considering the complexity of the overlap of cliques, we propose a fast verification algorithm by constructing a smaller flow network based on $G[T]$. Based on the fact that only the h-cliques at the boundary of $G[T]$ affect h-clique compact numbers in $G[T]$ compared to the h-clique compact numbers in G, we use a set P to record these h-cliques. For each $\psi_h^{P_r} \in P$, the number of vertices that contained both in $\psi_h^{P_r}$ and $G[T]$ is cnt_{P_r}. The flow network $\mathcal{F}(V_{\mathcal{F}}, E_{\mathcal{F}})$ is shown in Figure 6. The vertex set of $\mathcal{F}$ is $\{s\} \cup V \cup \Psi_h \cup P \cup \{t\}$. We add the boundary h-clique set P into $V_{\mathcal{F}}$ to ensure that the results of solving the flow network of $G[T]$ are precisely consistent with that of G. The arc set of $\mathcal{F}$ is given as follows. The arcs for vertices and h-cliques in $G[T]$ are the same as the former flow network. For each $\psi_h^{P_r} \in P$, we add cnt_{P_r} incoming arcs of capacity $1 + \frac{h-cnt_{P_r}}{cnt_{P_r}}$ from the vertices that both in $\psi_h^{P_r}$ and $G[T]$, and cnt_{P_r} outgoing arcs of capacity of $h-1$ to the same set of vertices.

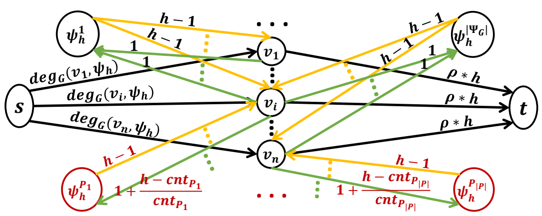


Figure 6: The flow network of `DeriveCompact`(G, ρ, P)

In Algorithm 5, the h-clique density of $G[S]$ is assigned to ρ (Line 1). Then, a breadth-first search is performed. U is used to store the vertices to be traversed (Line 4). The first vertex v from U is popped out (Line 6), and all h-cliques containing v are iterated (Lines 7-25). For each ψ_h that not in W, if any vertex w in ψ_h that $\overline{\phi}_h(w) < \rho$, ψ_h will not affect the h-clique compact number of w (Lines 9-13); if any vertex w is in any outputted LhCDS, False is assigned to VerifyLhCDS (Lines 17-18); the number of vertices in ψ_h that $\underline{\phi}_h(w) \leq \rho$ is recorded and ψ_h is added into P (Lines 19-25). All neighbors of v are iterated (Lines 26-30). For each neighbor w that not in T, if $\underline{\phi}_h(w) > \rho$, False is assigned toVerifyLhCDS (Line 28). If $\underline{\phi}_h(w) \leq \rho < \overline{\phi}_h(w)$, w will be added into U and T (Line 30). If VerifyLhCDS is False, a subgraph $G[T]$ induced by T and peripheral h-cliques in P are used to compute all h-clique ρ-compact subgraphs in $G[T]$ via min-cut (Line 32). Finally, True is returned if $G[S]$ is maximal h-clique ρ-compact; otherwise, the algorithm returns False (Line 33). The flow network here is much smaller.

Theorem 5. *Given a graph G and a self-densest subgraph $G[S]$, $G[S]$ is an LhCDS of G if and only if the fast LhCDS verification algorithm returns True.*

Proof. On the one hand, if $G[S]$ is an LhCDS of G, $G[S]$ is still an LhCDS in $G[T]$, because only the h-cliques in P might increase the h-clique compact numbers in $G[T]$ compared to the h-clique compact numbers in G. Otherwise, there exists a vertex v with h-cliques in P contained in the maximal h-clique $d_{\psi_h}(G[S])$-compact subgraph containing $G[S]$, and we can construct a larger h-clique $d_{\psi_h}(G[S])$-compact subgraph in G by adding vertices with $\underline{\phi}_h(w) > d_{\psi_h}(G[S])$ connected to v, which contradicts that $G[S]$ is an LhCDS. On the other hand, if $G[S]$ is not an LhCDS of G, we will find a larger h-clique $d_{\psi_h}(G[S])$-compact subgraph containing $G[S]$ in $G[T]$. Therefore, $G[S]$ is not an LhCDS in $G[T]$, and the algorithm returns False. Therefore, the algorithm return True only when $G[S]$ is an LhCDS of G. □

Algorithm 5: The fast LhCDS verification algorithm

Input: $G = (V, E)$, S, $\Psi_h(G)$, $\overline{\phi}_h$, $\underline{\phi}_h$
Output: VerifyLhCDS
1 $\rho \leftarrow d_{\psi_h}(G[S])$, VerifyL$h$CDS $\leftarrow$ True, Valid $\leftarrow$ True;
2 $U \leftarrow$ an empty queue, $P \leftarrow \emptyset$, $T \leftarrow \emptyset$, $W \leftarrow \emptyset$, $cnt \leftarrow 0$;
3 **foreach** $u \in S$ **do**
4   **if** $u \notin T$ **then** push u to U, insert u into T ;
5   **while** *U is not empty* **do**
6         $v \leftarrow$ pop out the front vertex in U;
7   **foreach** $\psi_h \in \Psi_h(G)$ *where* $v \in \psi_h$ **do**
8   Valid $\leftarrow$ True;
9   **if** $\psi_h \notin W$ **then**
10   **foreach** $w \in \psi_h$ **do**
11   **if** $\overline{\phi}_h(w) < \rho$ **then** Valid $\leftarrow$ False ;
12   insert ψ_h into W;
13   **else** Valid $\leftarrow$ False ;
14   **if** *Valid* **then**
15                 $cnt \leftarrow 1$;
16   **foreach** $w \in \psi_h$ *and* $w \neq v$ **do**
17   **if** *$w \notin T$ and w is in any LhCDS* **then**
18   VerifyLhCDS $\leftarrow$ False;
19   **if** $\underline{\phi}_h(w) \leq \rho$ **then**
20   **if** $w \notin T$ **then**
21   push w to U, insert w into T;
22                         $cnt \leftarrow cnt + 1$;
23   **if** $cnt \neq h$ *and* $\psi_h \notin P$ **then**
24   insert ψ_h and cnt into P;
25   VerifyLhCDS $\leftarrow$ False;
26   **foreach** $(v, w) \in E$ **do**
27   **if** $w \notin T$ *and* $\underline{\phi}_h(w) > \rho$ **then**
28   VerifyLhCDS $\leftarrow$ False;
29   **else if** $w \notin T$ *and* $\overline{\phi}_h(w) > \rho$ **then**
30   push w to U, add w into T;
31 **if** *VerifyLhCDS* **then** **return** True ;
32 $G' \leftarrow$ `DeriveCompact` $(G[T], \rho - \frac{1}{|V(G[T])|^2}, P)$;
33 **return** $G[S]$ is a connected component in G';

4.5 The LhCDS Discovery Algorithm (IPPV)

Combining all the algorithms above, we derive the LhCDS discovery algorithm, called the `IPPV` algorithm shown in Algorithm 6. An empty stack st is initialized, and G' is assigned to G (Line 1). The bounds of h-clique compact numbers are initialized via `InitializeBd` (Line 2). LhCDS candidates are derived via `ProposeLC` and Prune (Line 4-5). Next, the LhCDS candidates in $\mathcal{S}$ are reversely pushed into st, and the first LhCDS candidate in st, the one with the highest ϕ_h value, is popped out (Lines 6-7). The LhCDS candidate is verified by IsDensest (Line 8) and VerifyLhCDS (line 9). If $G[S]$ is an LhCDS, it will be outputted, and k is decreased by 1 (Line 10). If $G[S]$ is not an LhCDS but is self-densest, S is updated as the top LhCDS candidate from st (Line 12). Then, $G[S]$ is assigned to G' for the next iteration (Line 13). The above process is repeated until top-k LhCDSes are found (line 3) or the stack is empty (Line 11). Our algorithms can also be extended to find all LhCDSes.

Algorithm 6: The iterative propose-prune-and-verify algorithm based on convex programming (IPPV)

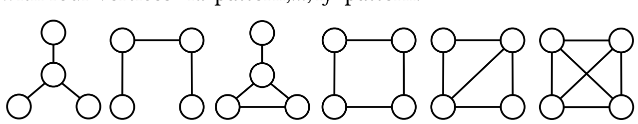
Input: $G = (V, E)$, number of iterations T, a integer k
Output: top-k LhCDS

```
1  st ← an empty stack; G' ← G;
2  φ̄_h, φ_h ← InitializeBd(G', h);
3  while k > 0 do
4      S, φ̄_h, φ_h ← ProposeLC (G', T, φ̄_h, φ_h) ;
5      S ← Prune (G, S, φ̄_h, φ_h);
6      foreach S ∈ S reversely do push S into st ;
7      S ← pop out the top stable group from st ;
8      if IsDensest (G[S]) then
9          if VerifyLhCDS (G, S, φ̄_h, φ_h) then
10             output G[S] ; k ← k − 1 ;
11         if st is empty then break ;
12         S ← pop out the top stable group from st ;
13     G' ← G[S] ;
```

Complexity Analysis. We use T to denote the number of iterations that SEQ-kClist++ needs. Each iteration of SEQ-kClist++ costs $O(n + |\Psi_h|)$. We use N_{CL} to represent the total number of LhCDS candidates, $N_{CL} \ll n$. Each iteration of verify an LhCDS candidate costs $O(n + |\Psi_h|)$. N_{Flow} is the number of times IsDensest and VerifyLhCDS are called. The time complexity of max-flow computation, which is $O((n + |\Psi_h|)^2 \cdot (n + |\Psi_h| \cdot h))$ for IsDensest and VerifyLhCDS when Dinic Algorithm is Applied. The time complexity of IPPV is $O((T+N_{CL})\cdot(n+|\Psi_h|)+N_{Flow}\cdot(n+|\Psi_h|)^2\cdot(n+|\Psi_h|\cdot h))$. The memory complexity is $(n + |\Psi_h|)$.

5 LhxPDS DISCOVERY

A pattern (also known as a motif) [15, 20, 37] is a small connected subgraph that appears frequently in a larger graph, which can be considered as a basic module. Figure 7 shows all kinds of patterns with four vertices: 4a-pattern,...,4f-pattern.



$(a)3-star$ $(b)4-path$ $(c)\ c3-star$ $(d)\ 4-loop$ $(e)\ 2-triangle$ $(f)\ 4-clique$

Figure 7: An example of all patterns with four vertices

We further show that the algorithm for the locally h-clique densest subgraph discovery problem can be extended to solve the locally general pattern densest subgraph discovery problem, which contributes to a deeper understanding of the organizational principles and functional modules within complex networks.

5.1 Densest Supermodular Set Decomposition

In this section, we discuss the reasonableness of extension from h-clique problem to general pattern problem. The convex programming of h-clique can be further generalized to the convex programming of supermodular sets, so that the convex programming for the general pattern densest subgraph problem and the corresponding compact number can be derived. A function $f : 2^V \to \mathbb{R}_+$ is said to be supermodular iff $\forall A, B \subseteq V, f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$. Harb et al. [14] proposed the densest supermodular subset (DSS) problem: given a normalized, nonnegative monotone supermodular function $f : 2^V \to \mathbb{R}_+$, return $S \subseteq V$ that maximizes $\frac{f(S)}{|S|}$. According to our observation, when $f(S) = |E(S)|$ and $f(S) = |\Psi_h(G[S])|$, the DSS problem is the DS and CDS problem, respectively. When $f(S)$ represents the number of a particular pattern in a graph, the problem is the densest problem of the proposed pattern. The convex program[14] for the densest supermodular set decomposition is $\text{CP}(G) := \min \left\{\sum_{u\in V} r(u)^2\right\}$, subject to: $r \in \left\{x \in \mathbb{R}^V | x \geq 0, x(S) \geq f(S) \text{ for all } S \subseteq V, x(V) = f(V)\right\}$.

With supermodularity, there is a property that each graph has a unique nested diminishingly decomposition for each type of density. The analysis of the generalization of CDS problem to DSS problem has triggered our thinking on the solution of locally general pattern densest problem.

5.2 Locally General Pattern Densest Subgraph Problem

Given an undirected graph $G = (V, E)$, $\psi_{hx}(V_{\psi_{hx}}, E_{\psi_{hx}})$ denotes a particular kind of pattern x with h vertices and $\Psi_{hx}(G)$ is the collection of the hx-patterns of G. $d_{\psi_{hx}}(G) = \frac{|\Psi_{hx}(G)|}{|V|}$ denotes the hx-pattern density of G. $deg_G(v, \psi_{hx})$ is the hx-pattern degree of v, i.e., the number of hx-patterns containing v. A graph $G = (V, E)$ is hx-pattern ρ-compact if and only if G is connected, and removing any subset of vertices $S \subseteq V$ will result in the removal of at least $\rho \times |S|$ hx-patterns in G. We can formally define a locally hx-pattern densest subgraph as follows.

Definition 7 (Locally hx-pattern densest subgraph (LhxPDS)). *A subgraph $G[S]$ of G is a locally hx-pattern densest subgraph of G if and only if there does not exist a supergraph $G[S']$ of $G[S]$ ($S' \supsetneq S$), such that $G[S']$ is also hx-pattern $d_{\psi_{hx}}(G)$-compact.*

Similarly, we formulate the locally hx-pattern densest subgraph problem as follows.

Definition 8 (Locally hx-pattern densest subgraph Problem (LhxPDS Problem)). *Given a graph G, an integer h, a pattern x and an integer k, the LhxPDS problem is to compute the top-k LhxPDSes ranked by the hx-pattern density in G.*

Here, we utilize our "iterative propose-prune-and-verify" pipeline to solve the LhxPDS problem. To apply the hx-pattern subgraph, there are some differences between Algorithm 6 and Algorithm 7 in the algorithmic details. In Algorithm 7, we need to counting hx-pattern graphs for Seq-kClist++ algorithm and derive candidate

Algorithm 7: The IPPV algorithm for LhxPDS

Input: $G = (V, E)$, number of iterations T, a integer k
Output: top-k LhxPDS

```
st ← an empty stack; G′ ← G;
φ̄_hx, φ_hx ← InitializeBd for hx-pattern (G′, h, x);
while k > 0 do
    S, φ̄_hx, φ_hx ← ProposeLC for hx-pattern (G′, T, φ̄_hx, φ_hx);
    S ← Prune for hx-pattern (G, S, φ̄_h, φ_h);
    foreach S ∈ S reversely do
        push S into st;
    S ← pop out the top stable group from st;
    if IsDensest (G[S]) for hx-pattern then
        if VerifyLhxPDS (G, S, φ̄_hx, φ_hx) then
            output G[S]; k ← k − 1;
        if st is empty then
            break;
        S ← pop out the top stable group from st;
    G′ ← G[S];
```

LhxPDS algorithm. In pruning part, the computation of hx-pattern graph cores is different for diverse kinds of patterns. Unlike h-clique, there may be more than one hx-pattern on a graph with h vertices. In verification part, the methods for reducing the size of subgraph to compute the min-cut need small adjustments for different patterns. In general, the process of extending our algorithm to general patterns is concise and clear.

6 EXPERIMENTS

6.1 Experimental Setup

The datasets we use are undirected real-world graphs [19, 29], including social networks, biological networks, web graphs, and collaboration networks. All datasets are listed in Table 2.

Table 2: Datasets used in our experiments

Name	Abbr.	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert\Psi_3\rvert$	$\lvert\Psi_5\rvert$
soc-hamsterster	HA	2,426	16,630	53,251	298,013
CA-GrQc	GQ	5,242	14,484	48,260	2,215,500
fb-pages-politician	PP	5,908	41,706	174,632	2,002,250
fb-pages-company	PC	14,113	52,126	56,005	207,829
web-webbase-2001	WB	16,062	25,593	21,115	382,674
CA-CondMat	CM	23,133	93,439	173,361	511,088
soc-epinions	EP	26,588	100,120	159,700	521,106
Email-Enron	EN	36,692	183,831	727,044	5,809,356
loc-gowalla	GW	196,591	950,327	2,273,138	14,570,875
DBLP	DB	317,080	1,049,866	2,224,385	262,663,639
Amazon	AM	334,863	925,872	667,129	61,551
soc-youtube	YT	495,957	1,936,748	2,443,886	5,306,643
soc-lastfm	LF	1,191,805	4,519,330	3,946,207	10,404,656
soc-flixster	FX	2,523,386	7,918,801	7,897,122	96,315,278
soc-wiki-talk	WT	2,394,385	4,659,565	9,203,519	382,777,822

We compare the performances of the following algorithms:

`IPPV` : the top-k LhCDS discovery algorithm proposed by us.

`LTDS` [31] : the top-k LTDS discovery algorithm based on the maximum-flow, which solves the LhCDS problem with $h = 3$.

`Greedy` : the top-k CDS discovery algorithm based on `KClist++` [33] using greedy approach. It has no guarantee on the locally densest property.

All algorithms are implemented in C++ and compiled by g++ compiler at -O3 optimization level. All experiments are evaluated on a machine with Intel(R) Xeon(R) CPU 3.20GHz processor and 128GB memory, with Ubuntu operating system. Algorithms running for more than 48 hours will be forcibly terminated.

6.2 Efficiency

In this section, we conduct experimental analysis on the efficiency of algorithms and summarize the influence of different parameter changes on the running time.

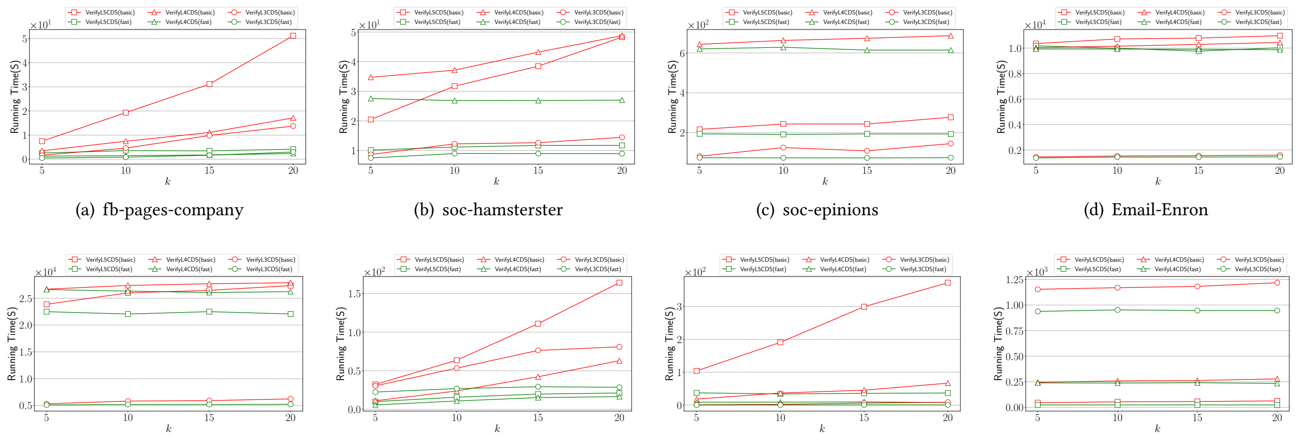
6.2.1 Efficiency: `IPPV` *vs* `LTDS`*.* Since LTDS is L3CDS, $h = 3$. We set $k = 5$ to observe the running time of the two algorithms in Table 3. We compare `IPPV` with `LTDS` on all datasets, and there are significant efficiency improvements on all datasets. The main bottleneck of LTDS is the time-consuming verification part, and the reason is that the upper and lower bounds of `LTDS` are not as tight as that of `IPPV`, so there will be more failures in the verification part, which demonstrates the superiority of our propose-prune-and-verify pipeline. The running time of our algorithm is closely related to the size of the graph and the number of h-cliques. A rise in the number of h-cliques can cause an increase in running time.

Table 3: Efficiency of IPPV and LTDS

Dataset	IPPV	LTDS	Speedup
soc-hamsterster	**7.50(s)**	46.54	6.20×
CA-GrQc	**0.38**	18.97	49.92 ×
fb-pages-politician	**32.32**	436.30	13.50×
fb-pages-company	**2.56**	51.48	20.11×
web-webbase-2001	**0.14**	12.20	87.14×
CA-CondMat	**21.63**	541.63	25.04×
soc-epinions	**82.54**	558.91	6.77×
Email-Enron	**1369.84**	2253.14	1.64×
loc-gowalla	**5095.63**	68216.14	13.39×
DBLP	**360.49**	4888.93	13.56×
Amazon	**1118.08**	1308.53	1.17×
soc-youtube	**9070.89**	42821.99	4.72×
soc-lastfm	**11223.13**	≥172, 800	≥ 15.40 ×
soc-flixster	**3018.62**	≥ 172, 800	≥ 57.24 ×
soc-wiki-talk	**57382.42**	≥ 172, 800	≥ 3.011 ×

6.2.2 Efficiency improvement by fast verification algorithm. We use VerifyLhCDS(basic) to represent the `IPPV` algorithm with Algorithm 4 and VerifyLhCDS(fast) to represent the `IPPV` algorithm with Algorithm 5. Their running times are compared in Figure 8. The fast verification algorithm with a smaller flow network is much faster than basic verification method. Especially as k increases, the efficiency gap between the two algorithms becomes more apparent. We also compare the running time of the two verification algorithms in the total running time in Figure 9, and the acceleration effect of the fast algorithm is obvious. The results demonstrate the importance and benefit of optimizing the verification algorithm.

6.2.3 the running time trends with k. Parameter k has a more pronounced impact on the running time of the algorithm than h. The experiments in Figure 8 indicate a direct relationship, where an increase in k corresponds to a proportional increase in execution time. This trend is consistently observed across different datasets,



(a) fb-pages-company (b) soc-hamsterster (c) soc-epinions (d) Email-Enron

(e) loc-gowalla (f) CA-CondMat (g) CA-GrQc (h) Amazon

Figure 8: Running time of algorithms with different h (= 3,4,5) and k. Red is VerifyLhCDS(basic), Green is VerifyLhCDS(fast)

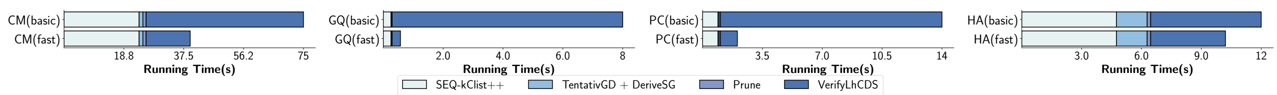
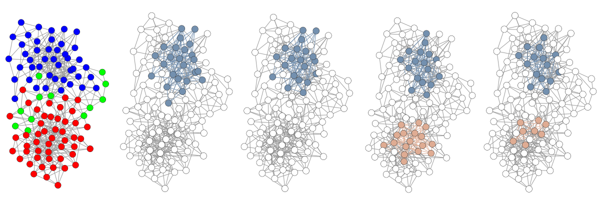


Figure 9: Running time of each part of IPPV with $h = 3$ and $k = 20$

which strengthens the premise that k is an important factor in computational complexity. The running time of both algorithms increases significantly for incremental values of k. The only deviation is observed in the Email-Enron dataset, where the running time remain relatively static despite changes in k, due to the fact that the total number of LhCDSes in this dataset is smaller than k.

6.2.4 The running time trends with h. We took $h = 3, 4, 5$ to compare the impact of h on the running time. The results are shown in Figure 8. When $h = 5$, the running time is generally longer. The reason is that when $h = 5$, the number of 5-clique is larger, as shown in Table 2. On Amazon, the running time is shorter when $h = 5$, because the number of 5-cliques is smaller. The running time is proportional to the number of h-cliques with different h.



(a) categories (b) 2-clique (c) 3-clique (d) 4-clique (e) 5-clique

Figure 10: LhCDS case study on real network (the top-1 LhCDS: steelblue; the top-2 LhCDS: orange vertices)

6.3 Comparison of Subgraphs

We visualize the result LhCDSes with different h. Figure 10 shows a network of books about US politics which were sold by Amazon [17]. The vertices represent different books, which fall into neutral(green), liberal(blue), and conservative(red) categories. The edges represent frequent co-purchasing of books by the same buyers, which indicate "customers who bought this book also bought the other books" feature on Amazon. The set of steelblue vertices is the top-1 LhCDS, and if exists, the set of orange vertices is the top-2 LhCDS of the h-clique. As shown in Figure 10, LhCDSes with larger h are closer to a clique. Besides, when h is larger, LhCDSes can find multiple dense communities in different fields: L4CDSes contain both liberal and conservative book communities, whereas LDSes only contain liberal book community.

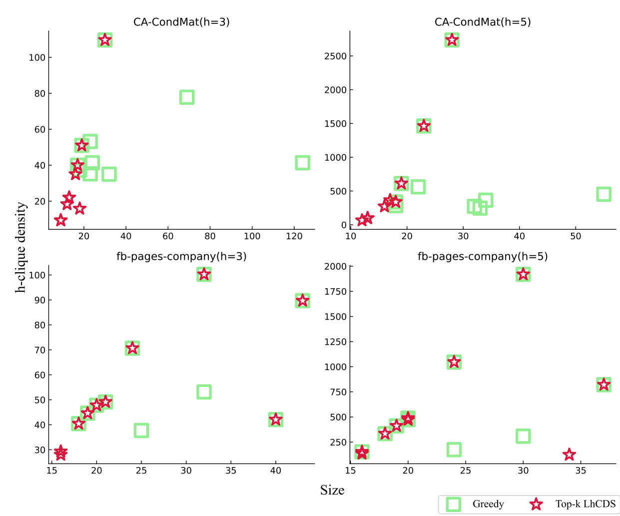


Figure 11: Subgraph statistics of h-density and size

Next, we compare the LhCDSes detected by our algorithm and the h-clique densest subgraphs found by the `Greedy` algorithm. We select $h = 3, 5$ on two datasets, respectively, and the results are shown in Figure 11. First, the results of the two algorithms overlap to a certain extent, among which the top-1 CDS is the same because the first LhCDS must be the h-clique densest subgraph in the whole graph. Second, there is a certain difference between the returned subgraphs of the `Greedy` algorithm and `IPPV` because the returned subgraphs of `Greedy` do not ensure the locally densest property. Therefore, the `Greedy` algorithm can not solve the LhCDS problem well. The two algorithms totally overlap if and only if the top-k h-clique densest subgraph belongs to different regions occasionally.

6.4 Clustering Coefficient of Different h

Since near clique is an important criterion for evaluating dense subgraphs, we evaluate how LhCDSes of different h are close to the clique structure. In graph theory, *clustering coefficient* is a measure of the degree to which vertices in a graph tend to cluster together, which is a direct measure to the degree of near clique. For each vertex $u \in V$, which has k_u neighbors N_u ($|N_u| = k_u$), the clustering coefficient of u is $C_u = \frac{2|\{e_{vw}:v,w \in N_u, e_{vw} \in E\}|}{k_u(k_u-1)}$. We compare the average C_u of all the LhCDSes of different h in Table 4.

Table 4: Average Clustering coefficient of different h values

dataset	Average Clustering coefficient				
	h=2	h=3	h=5	h=7	h=9
fb-pages-company	0.582	0.852	0.895	0.915	**0.930**
soc-hamsterster	0.480	0.910	0.990	0.984	**0.995**
fb-pages-politician	0.583	0.683	0.776	0.798	**0.835**
CA-CondMat	0.567	0.977	**0.992**	0.992	0.991
soc-epinions	0.231	0.722	0.701	0.705	**0.773**
web-webbase-2001	0.831	0.884	0.989	**0.992**	0.979
CA-GrQc	0.533	0.975	0.982	**0.985**	OOM

According to the results shown in Table 4, when h is larger, the average C_u is generally larger, showing LhCDSes with larger h are closer to clique. In addition, there is a big difference between $h = 3$ and $h = 2$ (L2CDS is LDS), which shows that LDS is less dense than other LhCDS. Our algorithm is important for finding near-clique subgraphs, which can not be replaced by LDS.

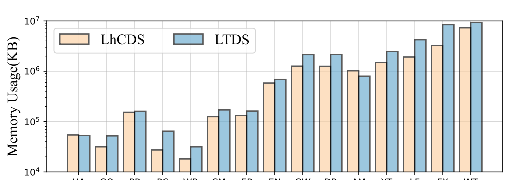


Figure 12: Memory usage of algorithms

6.5 Memory Overheads

We compare the memory utilization for the `IPPV` and `LTDS` algorithms across all datasets ($h = 3$). Figure 12 illustrates a clear correlation between memory usage and dataset size, with $k = 5$. `IPPV` algorithm strategically reduces the size of candidate subgraphs through a pruning mechanism prior to evaluating self-compactness. The verifying part often dominates memory consumption.

6.6 The number of iterations

To choose the optimal number of iterations T of `SEQ-kClist++`, we set different T on the `IPPV` algorithm. We select $T = 5, 10, 15, 20, 40, 60, 80, 100$ respectively, as shown in Figure 13. The experiment on eight datasets shows that the optimal performance is between 15 and 20 iterations. In our experiments, we choose $T = 20$.

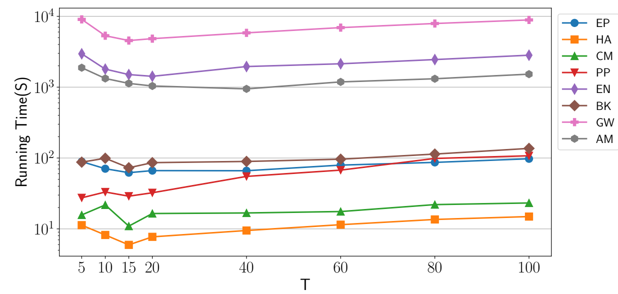
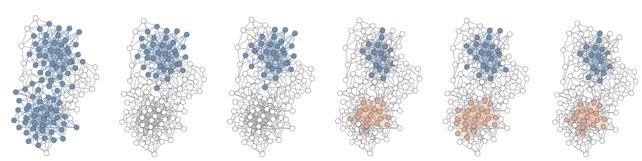


Figure 13: The running time of four datasets with different T

6.7 Case Study of LhxPDS

We utilize the same real dataset [17] to experimentally illustrate the LhxPDS problem. For each pattern depicted in Figure 7, we compute the results of L4xPDS. In Figure 14, the set of steelblue vertices is the top-1 LhxPDS, and if exists, the set of orange vertices is the top-2 LhxPDS of the pattern hx. It is evident that the L4xPDS corresponding to various patterns exhibit differences in terms of the number of L4xPDS, the number of vertices, and the position of vertices. The L4xPDSes of different patterns 4x correspond to the respective solutions of different problems. To delve deeper into graph analysis, tasks such as community clustering can be extended to explore the L4xPDS subgraph.



(a) 3-star (b) 4-path (c) c3-star (d) 4-loop (e) 2-triangle (f) 4-clique

Figure 14: L4xPDS case study on real network (the top-1 LhxPDS: steelblue; the top-2 LhxPDS: orange vertices)

7 CONCLUSION

In this paper, we study how to discover locally h-clique densest subgraphs in a graph G ,i.e., the LhCDS problem. We present an iterative propose-prune-and-verify pipeline for top-k LhCDS detection. The h-clique compact number and graph decomposition method to propose LhCDS candidates more efficiently is proposed. A new optimized verification algorithm is designed, and its correctness is proved. The extension of our algorithm to solve the locally general pattern densest subgraph problem is feasible and promising. Extensive experiments on real datasets show the high efficiency and scalability of our proposed algorithm. In the future, we will continue to optimize the algorithm of LhCDS problem and further explore the LhxPDS problem.

REFERENCES

[1] Albert Angel, Nick Koudas, Nikos Sarkas, Divesh Srivastava, Michael Svendsen, and Srikanta Tirthapura. 2012. Dense subgraph maintenance under streaming edge weight updates for real-time story identification. *The VLDB Journal* 23 (2012), 175–199. https://api.semanticscholar.org/CorpusID:2185310

[2] Bahman Bahmani, Ravi Kumar, and Sergei Vassilvitskii. 2012. Densest Subgraph in Streaming and MapReduce. *ArXiv* abs/1201.6567 (2012).

[3] Austin R. Benson, David F. Gleich, and Jure Leskovec. 2016. Higher-order organization of complex networks. *Science* 353 (2016), 163 – 166. https://api.semanticscholar.org/CorpusID:3635447

[4] Digvijay Boob, Yu Gao, Richard Peng, Saurabh Sawlani, Charalampos E. Tsourakakis, Di Wang, and Junxing Wang. 2019. Flowless: Extracting Densest Subgraphs Without Flow Computations. *Proceedings of The Web Conference 2020* (2019).

[5] Moses Charikar. 2000. Greedy approximation algorithms for finding dense components in a graph. In *International Workshop on Approximation Algorithms for Combinatorial Optimization*.

[6] Chandra Chekuri, Kent Quanrud, and Manuel R. Torres. 2022. Densest Subgraph: Supermodularity, Iterative Peeling, and Flow. In *ACM-SIAM Symposium on Discrete Algorithms*.

[7] Jie Chen and Yousef Saad. 2012. Dense Subgraph Extraction with Application to Community Detection. *IEEE Transactions on Knowledge and Data Engineering* 24 (2012), 1216–1230. https://api.semanticscholar.org/CorpusID:11360561

[8] Maximilien Danisch, T-H. Hubert Chan, and Mauro Sozio. 2017. Large Scale Density-friendly Graph Decomposition via Convex Programming. *Proceedings of the 26th International Conference on World Wide Web* (2017).

[9] Yixiang Fang, Kaiqiang Yu, Reynold Cheng, Laks V. S. Lakshmanan, and Xuemin Lin. 2019. Efficient Algorithms for Densest Subgraph Discovery. *Proc. VLDB Endow.* 12 (2019), 1719–1732.

[10] Adriano Fazzone, Tommaso Lanciano, Riccardo Denni, Charalampos E Tsourakakis, and Francesco Bonchi. 2022. Discovering polarization niches via dense subgraphs with attractors and repulsers. *Proceedings of the VLDB Endowment* 15, 13 (2022), 3883–3896.

[11] David Gibson, Ravi Kumar, and Andrew Tomkins. 2005. Discovering Large Dense Subgraphs in Massive Graphs. In *Very Large Data Bases Conference*. https://api.semanticscholar.org/CorpusID:120822

[12] A. Gionis, Flavio Paiva Junqueira, Vincent Leroy, Marco Serafini, and Ingmar Weber. 2013. Piggybacking on Social Networks. *Proc. VLDB Endow.* 6 (2013), 409–420. https://api.semanticscholar.org/CorpusID:2240241

[13] Andrew V. Goldberg. 1984. Finding a Maximum Density Subgraph. In *Technical report,University of California, Berkeley*.

[14] Elfarouk Harb, Kent Quanrud, and Chandra Chekuri. 2022. Faster and Scalable Algorithms for Densest Subgraph and Decomposition. In *Neural Information Processing Systems*.

[15] Jiafeng Hu, Reynold Cheng, Kevin Chen-Chuan Chang, Aravind Sankar, Yixiang Fang, and Brian Yee Hong Lam. 2019. Discovering Maximal Motif Cliques in Large Heterogeneous Information Networks. *2019 IEEE 35th International Conference on Data Engineering (ICDE)* (2019), 746–757. https://api.semanticscholar.org/CorpusID:174819659

[16] Ruoming Jin, Yang Xiang, Ning Ruan, and David Fuhry. 2009. 3-HOP: a high-compression indexing scheme for reachability query. *Proceedings of the 2009 ACM SIGMOD International Conference on Management of data* (2009). https://api.semanticscholar.org/CorpusID:7625491

[17] Valdis Krebs. 2004. Books about US politics. Unpublished. http://www.orgnet.com/

[18] Tommaso Lanciano, Atsushi Miyauchi, Adriano Fazzone, and Francesco Bonchi. 2023. A survey on the densest subgraph problem and its variants. *arXiv preprint arXiv:2303.14467* (2023).

[19] Jure Leskovec and Andrej Krevl. 2014. SNAP Datasets: Stanford Large Network Dataset Collection. http://snap.stanford.edu/data.

[20] Jure Leskovec, Ajit Singh, and Jon M. Kleinberg. 2006. Patterns of Influence in a Recommendation Network. In *Pacific-Asia Conference on Knowledge Discovery and Data Mining*. https://api.semanticscholar.org/CorpusID:332896

[21] Ruiming Li, Jung-Yu Lee, Jinn-Moon Yang, and Tatsuya Akutsu. 2022. Densest subgraph-based methods for protein-protein interaction hot spot prediction. *BMC Bioinformatics* 23 (2022).

[22] Yike Liu, Tara Safavi, Abhilash Dighe, and Danai Koutra. 2018. Graph summarization methods and applications: A survey. *ACM computing surveys (CSUR)* 51, 3 (2018), 1–34.

[23] Wensheng Luo, Chenhao Ma, Yixiang Fang, and Laks VS Lakshman. 2023. A Survey of Densest Subgraph Discovery on Large Graphs. *arXiv preprint arXiv:2306.07927* (2023).

[24] Chenhao Ma, Reynold Cheng, Laks VS Lakshmanan, and Xiaolin Han. 2022. Finding locally densest subgraphs: a convex programming approach. *Proceedings of the VLDB Endowment* 15, 11 (2022), 2719–2732.

[25] Michael Mitzenmacher, Jakub W. Pachocki, Richard Peng, Charalampos E. Tsourakakis, and Shen Chen Xu. 2015. Scalable Large Near-Clique Detection in Large-Scale Networks via Sampling. *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (2015).

[26] Gergely Palla, Imre Derényi, Illés J. Farkas, and Tamás Vicsek. 2005. Uncovering the overlapping community structure of complex networks in nature and society. *Nature* 435 (2005), 814–818. https://api.semanticscholar.org/CorpusID:3250746

[27] Jean-Claude Picard and Maurice Queyranne. 1982. A network flow solution to some nonlinear 0-1 programming problems, with applications to graph theory. *Networks* 12 (1982), 141–159.

[28] Lu Qin, Rong-Hua Li, Lijun Chang, and Chengqi Zhang. 2015. Locally densest subgraph discovery. In *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. 965–974.

[29] Ryan A. Rossi and Nesreen K. Ahmed. 2015. The Network Data Repository with Interactive Graph Analytics and Visualization. In *AAAI*. https://networkrepository.com

[30] Barna Saha, Allison Hoch, Samir Khuller, Louiqa Raschid, and Xiao-Ning Zhang. 2010. Dense Subgraphs with Restrictions and Applications to Gene Annotation Graphs. In *Annual International Conference on Research in Computational Molecular Biology*. https://api.semanticscholar.org/CorpusID:11280177

[31] Raman Samusevich, Maximilien Danisch, and Mauro Sozio. 2016. Local triangle-densest subgraphs. In *2016 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM)*. IEEE, 33–40.

[32] Victor Spirin and Leonid A. Mirny. 2003. Protein complexes and functional modules in molecular networks. *Proceedings of the National Academy of Sciences of the United States of America* 100 (2003), 12123 – 12128. https://api.semanticscholar.org/CorpusID:136093

[33] Bintao Sun, Maximilien Danisch, TH Hubert Chan, and Mauro Sozio. 2020. Kclist++: A simple algorithm for finding k-clique densest subgraphs in large graphs. *Proceedings of the VLDB Endowment (PVLDB)* (2020).

[34] Tran Ba Trung, Lijun Chang, Nguyen Tien Long, Kai Yao, and Huynh Thi Thanh Binh. 2023. Verification-Free Approaches to Efficient Locally Densest Subgraph Discovery. *2023 IEEE 39th International Conference on Data Engineering (ICDE)* (2023), 1–13. https://api.semanticscholar.org/CorpusID:260171546

[35] Charalampos Tsourakakis. 2015. The k-clique densest subgraph problem. In *Proceedings of the 24th international conference on world wide web*. 1122–1132.

[36] Charalampos E. Tsourakakis, Francesco Bonchi, A. Gionis, Francesco Gullo, and Maria A. Tsiarli. 2013. Denser than the densest subgraph: extracting optimal quasi-cliques with quality guarantees. *Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining* (2013). https://api.semanticscholar.org/CorpusID:213308

[37] Stefan Wuchty, Zoltán N. Oltvai, and A L Barabasi. 2003. Evolutionary conservation of motif constituents in the yeast protein interaction network. *Nature Genetics* 35 (2003), 176–179. https://api.semanticscholar.org/CorpusID:1627007

[38] Long Yuan, Lu Qin, Xuemin Lin, Lijun Chang, and W. Zhang. 2015. Diversified top-k clique search. *The VLDB Journal* 25 (2015), 171–196. https://api.semanticscholar.org/CorpusID:15668109

[39] Feng Zhao and Anthony Kum Hoe Tung. 2012. Large Scale Cohesive Subgraphs Discovery for Social Network Visual Analysis. *Proc. VLDB Endow.* 6 (2012), 85–96. https://api.semanticscholar.org/CorpusID:12588941

[40] Zhaonian Zou. 2013. Polynomial-time algorithm for finding densest subgraphs in uncertain graphs. In *Proceedings of MLG Workshop*.