EMI: An Efficient Algorithm for Identifying Maximal Rigid Clusters in 3D Generic Graphs

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***Abstract*— Identifying the Maximal Rigid subGraphs (MRGs) whose relative formations cannot deform continuously in $\Re^d$, is a fundamental problem in network formation control and network localization. When $d = 3$, it becomes extremely challenging and has been open for decades because the fundamental Laman condition doesn't hold in $\Re^3$. This paper presents a new understanding of this problem. Because of the existence of "implicit hinges" in 3D, its essence should be to detect the Maximal Rigid Clusters (MRCs). An MRC is a maximal set of vertices in which each vertex is mutually rigid to the others, but the vertices are not necessarily connected. We show that the MRGs in the original graph can be easily deduced from the connected components generated by the MRCs. For efficiently identifying the MRCs, at first, a randomized algorithm to detect mutually rigid vertex pairs is exploited. Based on this, a *Basic MRC Identification algorithm (BMI)* is proposed, which is an exact algorithm that can detect all MRCs based on the extracted rigid vertex pairs, but it has $O(|V|^4)$ time complexity. To further pursue an efficient algorithm, we observe the "hinge MRCs" appear rarely. So an *Efficient framework for MRC Identification (EMI)* is proposed. It consists of two steps: 1) a *Trimmed-BMI* algorithm that guarantees to detect all simple MRCs and may miss only hinge MRCs; 2) a *Trim-FIX* algorithm that can find all hinge MRCs. We prove EMI can guarantee to detect all the MRCs as accurately as *BMI*, using $O(|V|^3)$ times. Further, we show *EMI* achieves magnitudes of times faster than *BMI* in experiments. Extensive evaluations verify the effectiveness and high efficiency of *EMI* in various 3D networks. We have uploaded the code of the related program to https://github.com/fdwqh/EMI-algorithm.**

***Index Terms*— Maximum rigid cluster partition, 3D networks, rigid cluster, implicit hinge, mutual rigid pair.**

I. Introduction

RIGIDITY, in simple terms, is the property of a group of particles that always move as a whole, or the relative distance between the particles of this group remains unchanged during continuous motion [1]. Given a graph $G(V, E)$, the problem of identifying the maximal rigid subgraphs (MRGs) in G has attracted great research attentions[1], [2], which is a fundamental problem in network localization [3] and UAV network formation control etc. However, when the graph's configuration is considered in $\Re^3$, the MRG detection problem becomes difficult because the fundamental Laman theorem [4] doesn't hold in $\Re^3$. So the problem of identifying MRGs for graphs in $\Re^3$ remains open for decades to the best of our knowledge. But in real applications, the physical space is in $\Re^3$. The network localization problem and the UAV network formation problem all pose high requirements for efficient algorithms to solve the MRG detection problem in $\Re^3$.

This paper presents a new understanding of this problem. We differentiate the concepts of rigid cluster and rigid subgraph in G. A rigid cluster refers to a cluster of vertices that are mutually rigid to each other. The inter-distances among these nodes cannot deform continuously for edge constraints in the graph, but it doesn't require these vertices to form a connected subgraph. As the example shown in Fig.1, the vertices $\{1, 2, 3\}$ form a rigid cluster, but they are not connected. Whereas, rigid subgraph must be a connected subgraph. Existing studies [2], [5], [6], [7] detect rigid subgraphs by edge counting method based on Laman theorem [1]. We propose to detect rigid clusters using the original concept of infinitesimal rigidity ([8]), which is not restricted by edge counting. We then propose to detect the maximal rigid subgraphs (MRGs) based on the detected maximal rigid clusters (MRCs). More specifically, a Maximal Rigid subGraph (MRG) is a maximal connected component in G which is rigid and will not be rigid if any other neighboring vertex is added. A Maximal Rigid Cluster (MRC) is a maximal vertex set in G where the vertices are mutually rigid, and will not be rigid if any other vertex is added. We will show that MRGs are essentially the maximal rigid components in the graphs induced by the MRCs. We argue the essence should be to detect the MRCs because the MRGs can be easily deduced from the MRCs.

Further, the invalidity of Laman theorem in $\Re^3$ is mainly caused by the "implicit hinges" [9] in $\Re^3$ which doesn't appear in $\Re^2$. If two MRCs have only two vertices in common, the line segment generated by the two vertices forms a "hinge". The two MRCs can rotate around the hinge. If there isn't an edge between the two vertices, it is called an "implicit hinge". If all the implicit hinges in G are added into G, an augmented graph G_A will be formed. In G_A, Laman theorem is true and each MRC will correspond to a connected MRG in G_A. But in the original graph G, for the absence of the implicit hinges, the induced graph of MRCs may be disconnected and therefore non-rigid. MRGs are the connected components induced by the MRCs.

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